What is. . . a Smoke Ring?

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Outline

FIGURE: Vortex filaments in smoke. On the right, a big smoke ring produced by the vulcano 'Etna'.

Outline

1. The Vortex Filament Equation
2. The Non-Linear Schrödinger Equation
3. Invariants
4. Hashimoto Surfaces

Let γ be a closed curve, that moves with time. If γ evolves by the *vortex filament equation*

$$\dot{\gamma} = \gamma' \times \gamma'', \quad (VFE)$$

we call γ a *smoke ring*. Here the dot and the prime denote the derivative with respect to time and arclength, resp.

Some historical remarks:

- discovered by Da Rios (a student of Levi-Civita) in 1906
- independently rediscovered in 1965 by Betchov

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FIGURE: Smoke ring flow of a double eight. Source: GeometrieWerkstatt (Tuebingen).

Let γ be a closed Frenet curve parametrized by arclength. Its Frenet frame (T, N, B) is given by

$$T = \gamma',\ N = \frac{T'}{|T'|},\ B = T \times N.$$

The frame satisfies the following equations

$$T' = \kappa N,\ N' = -\kappa T + \tau B,\ B' = -\tau N.$$

κ is the curvature and τ the torsion of γ.

For Frenet curves the (VFE) can be rewritten as

$$\dot{\gamma} = \kappa B.$$

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The so called *Hashimoto-map* is given by the following formula:

$$q = \frac{1}{2}\kappa \exp\left(i \int \tau ds \right).$$

It relates the (VFE) to the non–linear cubic Schrödinger equation:

$$i\dot{q} + q'' + 2\left|q\right|^2 q = 0. \quad (NLSE)$$

Note: The relation is not restricted to the special case of Frenet curves. In general, one can use a special parallel frame: If κ_1 and κ_2 are the curvatures of the parallel frame, then the complex curvature function $q = \kappa_1 + i\kappa_2$ satisfies the (NLSE).

The (NLSE) is a well known completely integrable partial differential equation.
$\rightsquigarrow$ *infinite number of integrals of motion*

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The length of γ is preserved under the flow. Even more: The smoke ring moves without stretching.

$$\frac{d}{dt}|\gamma'|^2 = 2\langle\gamma', \dot{\gamma}'\rangle = 2\langle\gamma', (\gamma' \times \gamma'')'\rangle = 2\langle\gamma', \gamma' \times \gamma'''\rangle = 0.$$

From now on we will always assume that $\gamma\colon [0, L] \to \mathbb{R}^3$ is arclength parametrized for all time. We denote its length by L.

The total torsion T_γ and bending energy E_γ of γ are defined as follows:

$$T_\gamma = \int_0^L \tau \, ds, \quad E_\gamma = \int_0^L \kappa^2 \, ds.$$

The total torsion of a smoke ring is an invariant. We manipulate $\dot{\tau}$:

$$\begin{aligned}\dot{\tau} &= \langle N', B\rangle^{\cdot} = \langle \dot{N}', B\rangle + \langle N', \dot{B}\rangle = \langle \dot{N}', B\rangle + \langle N', \dot{T}\times N\rangle + \langle N', T\times \dot{N}\rangle \\ &= \langle \dot{N}, B\rangle' - \langle \dot{N}, B'\rangle + \langle \dot{T}, N\times N'\rangle + \langle \dot{N}, N'\times T\rangle \\ &= \langle \dot{N}, B\rangle' + \tau\langle \dot{N}, N\rangle + \langle \dot{T}, -\kappa N\times T + \tau N\times B\rangle + \langle \dot{N}, -\kappa T\times T + \tau B\times T\rangle \\ &= \langle \dot{N}, B\rangle' + \tau\langle \dot{N}, N\rangle + \langle \dot{T}, \kappa B + \tau T\rangle + \tau\langle \dot{N}, N\rangle,\end{aligned}$$

Since T and N are normalized we have $\langle \dot{T}, T\rangle = \langle \dot{N}, N\rangle = 0$. Further we get

$$\dot{T} = \dot{\gamma}' = \gamma' \times \gamma''' = T\times T'' = T\times\left(\kappa' N + \kappa N'\right) = \kappa' B - \kappa\tau N.$$

All together:

$$\dot{\tau} = \langle \dot{N}, B\rangle' + \langle \kappa' B - \kappa\tau N, \kappa B + \tau T\rangle = \langle \dot{N}, B\rangle' + \kappa\kappa' = \langle \dot{N}, B\rangle' + \tfrac{1}{2}\left(\kappa^2\right)'.$$

Since γ is closed, we obtain $\frac{d}{dt} T_\gamma = \int_0^L \dot{\tau}\, ds = 0$.

The bending energy of a smoke ring is an invariant. We want to manipulate $\left(\kappa^2\right)^{\cdot}$. First:

$$\begin{aligned}\dot{\kappa} &= \langle T', N\rangle^{\cdot} = \langle \dot{T}', N\rangle + \langle T', \dot{N}\rangle = \langle \left(\kappa' B - \kappa\tau N\right)', N\rangle + \langle T', \dot{N}\rangle \\ &= \langle \kappa' B' - (\kappa\tau)' N, N\rangle + \langle T', \dot{N}\rangle = \langle -\kappa'\tau N - (\kappa\tau)' N, N\rangle + \kappa\langle N, \dot{N}\rangle \\ &= -\kappa'\tau - (\kappa\tau)'.\end{aligned}$$

Hence

$$\left(\kappa^2\right)^{\cdot} = 2\dot{\kappa}\kappa = -2\left(\kappa'\tau + (\kappa\tau)'\right)\kappa = -2\left((\tau\kappa)\kappa' + (\tau\kappa)'\kappa\right) = -2\left(\tau\kappa^2\right)'.$$

Again, since γ is closed, we obtain $\frac{d}{dt}E_\gamma = \int_0^L \left(\kappa^2\right)^{\cdot} ds = 0$.

Another invariant is the area vector A_γ of γ. It is given by the following formula:

$$A_\gamma = \int_0^L \gamma \times \gamma' ds.$$

- A_γ encodes the projected area: for $v \in \mathbb{S}^2 \subset \mathbb{R}^3$ the area of the projection of γ onto the plane perpendicular to v is given by $\langle A_\gamma, v \rangle$

Differentiate A_γ with respect to t:

$$\begin{aligned} \frac{d}{dt} A_\gamma &= \int_0^L \frac{d}{dt} \gamma \times \gamma' ds = \int_0^L \dot{\gamma} \times \gamma' + \gamma \times \dot{\gamma}' ds = \int_0^L \dot{\gamma} \times \gamma' + (\gamma \times \dot{\gamma})' - \gamma' \times \dot{\gamma} ds \\ &= \int_0^L (\gamma \times \dot{\gamma})' - 2\gamma' \times \dot{\gamma} ds = \int_0^L (\gamma \times \dot{\gamma})' - 2\gamma' \times \left(\gamma' \times \gamma''\right) ds \\ &= \int_0^L (\gamma \times \dot{\gamma})' + 2\gamma'' ds = \int_0^L \left(\gamma \times \dot{\gamma} + 2\gamma'\right)' ds = 0. \end{aligned}$$

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FIGURE: Smoke ring flow of a double eight. Source: GeometrieWerkstatt (Tuebingen).

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FIGURE: Isospectral flow of a constant mean curvature torus. Source: GeometrieWerkstatt (Tuebingen).

Outline

Applying a certain transformation one obtains for each smoke ring an *associated family*.

- Sym–Bobenko formula gives explicit formulas for smoke rings of finite type

Matthias Heil (1995) developed tools for the numerical evalution of theta functions on Riemann surfaces, which led to nice visualizations.

The surfaces obtained by the flow of the smoke rings are called *Hashimoto surfaces*.

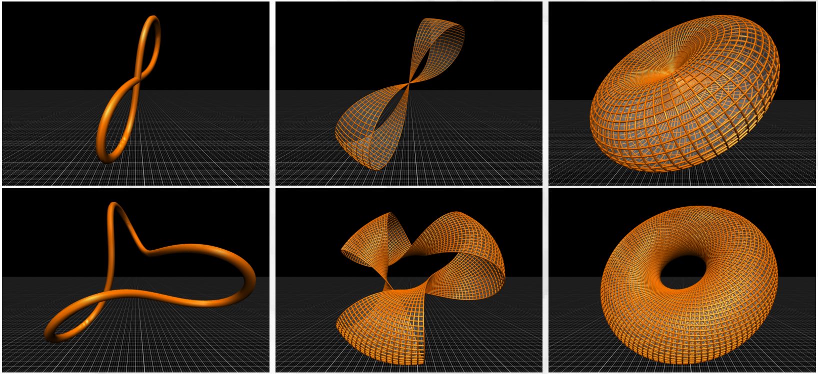

FIGURE: Double–periodic smoke rings of spectral genus 1.

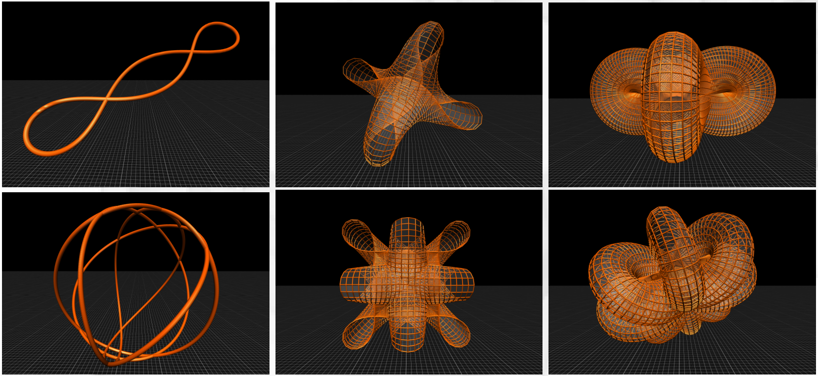

FIGURE: Double–periodic smoke rings of spectral genus 2.

LITERATURE

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